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Quantum distribution functions for radial observables

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Abstract. For quantum systems with two-dimensional configuration space we construct a physical radial momentum observable. Rescaling the radius we find that the dilation degrees of freedom form a Weyl algebra. With this we construct a radial Wigner quasiprobability distribution function.

1. Introduction

The Wigner quasiprobability distribution function is a familiar tool to many working in quantum and atom optics [1]. It is primarily used in the classical-quantum correspondence where the appearance of positive and negative regions of the Wigner function gives easily understood information concerning the probability concentrations and quantum interferences present within the quantum state [2]. Typically, one describes the Wigner function on a phase space which is labelled by the Cartesian coordinates of position and momentum. For physical systems which admit a two-dimensional cylindrical symmetry, for example, trapped ultracold ions, Bose condensates, etc, clearly a polar description of the Wigner function would be more natural. However, no such description has appeared in the literature. In this paper we show, in three stages, how a Wigner function for the radial observables can be constructed. This radial Wigner function could be reconstructed from experimental data much as recent experiments have reconstructed the Cartesian Wigner function for the one-dimensional motion of a trapped ion [3]. The angular parts of the complete four-dimensional Wigner function are complicated by the imposition of single-valuedness of the wavefunction under a rotation of 2π which cause the conjugate angular momentum to become discrete. We leave the angular part for a later work.

The stages towards the construction of a radial Wigner function proceed as follows. (1) A proper Wigner function possesses marginals which are true probability distributions for the observables whose eigenvalues parametrize the Wigner function and thus the phase space axes. For a single degree of freedom a mere transformation of the Cartesian position and momentum parameters into polar form does not yield a proper Wigner function for the polar parameters [4]. This is also true for higher-dimensional phase spaces. Central to the problem is a correct specification of the radial momentum operator. By noting the symmetry action of the momentum on the half-infinite radius operator we can construct a physical ‘conjugate’ momentum $\hat{P}^r$. This momentum is physical in that it is represented by a fully self-adjoint operator. The conjugate momentum found here is similar to those found in the method of geometric quantization [5].

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(2) Essential to the construction of the Wigner function is the existence of 'point' operators $\hat{A}(\lambda_1, \lambda_2)$, which obey, $\mathrm{Tr}[\hat{A}(\lambda_1, \lambda_2)\hat{A}^\dagger(\lambda_1', \lambda_2')] = \delta(\lambda_1 - \lambda_1')\delta(\lambda_2 - \lambda_2')$ [6, 14]. We find that the radial $\hat{A}(r, P^r)$, is not completely exponential in the radial position and momentum operators.

(3) Guided by the form of $\hat{A}(r, P^r)$ we make the operator transformation $\hat{v} \equiv \ln \hat{r}$. This is a well-defined transformation as $\hat{r}$ is a positive unbounded operator with a real non-negative spectrum. We then find, $[\hat{v}, \hat{P}^r] = \mathrm{i}\hbar$, the standard Heisenberg–Weyl algebra. Given this algebra and after rescaling the eigenbasis kets of $\hat{v}$ to recover the standard resolution of unity, we can now construct a radial Wigner function, $W(v, P^r)$. This radial (or dilation) Wigner function gives the proper marginal probability distributions for $\hat{v}$ and $\hat{P}^r$. Essentially, we have performed a unitary transformation whose corresponding classical counterpart is a canonical transformation from the polar coordinates $(r, \dot{r})$, to the new coordinates $(\ln r, \dot{r}/r)$, where the domains of the latter are fully infinite. From the Heisenberg–Weyl algebra for $\ln \hat{r}$ and $\hat{P}^r$, one can deduce the existence of a dilation ground state $|0\rangle_{\mathrm{dilations}}$. We then calculate the wavefunction of this ground state in the $\hat{r}$ basis. We finally calculate the radial Wigner function for the lowest Schwinger states $|l, 0\rangle$ (these are simultaneous eigenstates of energy and angular momentum), and briefly outline how, given a quantum state in a two-dimensional harmonic Fock representation, one can construct the radial reduced density matrix, and from there the radial Wigner function.

2. Radial momentum

Dirac, in his textbook on quantum mechanics [7], introduced the following momentum, conjugate to the radial coordinate

$$\hat{P}^D \equiv \frac{1}{\hat{r}}(\hat{x}\hat{P}_x + \hat{y}\hat{P}_y - \mathrm{i}\hbar/2). \tag{1}$$

The factor of 2 difference in the $\hbar$ term arises because we are working in two dimensions instead of the three in Dirac's case. Using the Cartesian commutation relations between $\hat{x}$ and $\hat{P}_x$ one can easily confirm that

$$[\hat{r}, \hat{P}^D] = \mathrm{i}\hbar. \tag{2}$$

Using the Campbell–Baker–Hausdorff (CBH) expansion and (2) we can show

$$\exp(\mathrm{i}\varsigma\hat{P}^D/\hbar)\hat{r}\exp(-\mathrm{i}\varsigma\hat{P}^D/\hbar) = \hat{r} + \varsigma. \tag{3}$$

Since $\hat{r}$ has a half-infinite spectrum, the operator $\hat{P}^D$ cannot be self-adjoint. This has been noted by many authors [8]. To discuss the adjoint properties of an operator we must be more precise and include the domain of an operator as part of that operator's definition. The adjoint of an operator $\hat{A}$, is defined to be $\hat{A}^\dagger$, such that $\hat{A}^\dagger\psi = \eta$ where $(\psi, \hat{A}\phi) = (\eta, \phi)$ for any ψ and all ϕ in the domain of $\hat{A}$ [9]. If $\hat{A}^\dagger = \hat{A}$ then A is said to be self-adjoint. For $\hat{A}$ to be self-adjoint it is crucial that the domains of $\hat{A}$ and $\hat{A}^\dagger$ be equal. The Dirac momentum (1), on the half-line $\mathbb{R}^+$ with measure $x\,\mathrm{d}x$, is not self-adjoint. To show this explicitly we must first obtain a representation of $\hat{P}^D$ in the $\hat{r}$ eigenket basis (i.e. $\hat{r}|r\rangle = r|r\rangle$). From the commutation relation (2) we must have $\langle r|\hat{P}^D|\psi\rangle \sim -\mathrm{i}\hbar(\partial_r + f(r))\langle r|\psi\rangle$. From (1), and expressing the Cartesian partial derivatives in polar coordinates we find

$$\langle r|\hat{P}^D|\psi\rangle = -\mathrm{i}\hbar\left(\partial_r + \frac{1}{2r}\right)\langle r|\psi\rangle. \tag{4}$$

Before discussing the adjointedness of this operator we must first examine whether $\hat{P}^D$ is Hermitian. We know that all observables in quantum mechanics are Hermitian self-adjoint operators. An operator $\hat{A}$ is said to be Hermitian if, for every ϕ and ψ in the domain of $\hat{A}$,

$$(A\psi, \phi) = (\psi, A\phi). \tag{5}$$

For the operator $-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$ defined to act on the whole real line, it is trivial to show (5) is satisfied as long as $\psi^*\phi|_{-\infty}^{\infty} = 0$. This is true since $\psi, \phi \in \mathcal{L}^2(\mathbb{R}, \mathrm{d}x)$. For the case at hand, however, it can easily be shown that if one allows the operator $\hat{P}^D$ to act on every normalizable wavefunction in $\mathcal{L}^2(\mathbb{R}^+, r\,\mathrm{d}r)$, then it will *not* be Hermitian. One has to restrict the domain of $\hat{P}^D$ by imposing a boundary condition at the origin. To see this most readily we rescale the wavefunctions to be $\psi \equiv V/\sqrt{r}$, and $\phi \equiv U/\sqrt{r}$, where now $U, V \in \mathcal{L}^2(\mathbb{R}^+, \mathrm{d}r)$, which gives $\lim_{r\to\infty} U(r) = 0$ and similarly for $V(r)$. Rewriting equation (5) in these rescaled wavefunctions for $\hat{A} = \hat{P}^D$, yields the condition $V^*U|_0^{\infty} = 0$. This condition is satisfied at $r = \infty$, however, from normalizability alone there are no conditions on the values of either U or V at the origin. Thus, to ensure that the operator $\hat{P}^D$ is Hermitian we must allow it to act only on a restricted class of all possible normalizable wavefunctions, i.e. to that class $\psi(r) \equiv U(r)/\sqrt{r}$ where $U(r) = 0$, at $r = 0$.

We now address the adjoint properties of the operator. An operator $\hat{A}$ may either be essentially self-adjoint, as is the operator $\hat{P}_x \equiv -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathbb{R}$, or may be self-adjoint on a restricted subclass of wavefunctions, as is the operator $\hat{P}_x$ on the interval $x \in [0, a]$, for the class of wavefunctions $\psi(x = 0) = \mathrm{e}^{\mathrm{i}\theta}\psi(x = a)$, or may possess *no* self-adjoint extensions. A method to determine whether an Hermitian operator $\hat{A}$ is either essentially self-adjoint, or possesses a self-adjoint extension, has been developed by von Neumann [20, 15, 21]. This method involves the determination of the *deficiency indices*, $n_\pm$, of the operator $\hat{A}$. If both $n_\pm = 0$, then the operator is essentially self-adjoint, while if $n_+ = n_-$, the operator possesses a self-adjoint extension. However, if $n_+ \neq n_-$, then no self-adjoint extension is possible and the operator, if Hermitian, cannot possibly represent any physical observable. The deficiency indices $n_\pm$, are the dimensions of the normalizable solution spaces of the equations $\hat{A}^\dagger\psi = \pm\mathrm{i}\psi$. For $\hat{A} = \hat{P}^D$, we obtain

$$-\mathrm{i}\hbar\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{2r}\right)\psi(r) = \pm\mathrm{i}\psi(r) \tag{6}$$

which, in the rescaled wavefunction $\psi \equiv U/\sqrt{r}$, becomes

$$-\mathrm{i}\frac{\mathrm{d}U}{\mathrm{d}r} = \pm\mathrm{i}U \tag{7}$$

and its solutions are $U(r) = U_{0\pm}\exp(\mp r)$. Each solution is parametrized by a single variable, however, the solution $\exp(+r)$ is not in $\mathcal{L}^2(\mathbb{R}^+, \mathrm{d}r)$, and thus we obtain the deficiency indices $(n_+, n_-) = (1, 0)$. Since these are not equal, *no* self-adjoint extension of the Dirac radial momentum exists.

Thus informed, we must abandon the use of the Dirac momentum and we will look instead for an operator which *will be self-adjoint without the need for an extension*. To do this we must find an operator which respects the half-infinite spectrum of $\hat{r}$ [20]. This can be achieved if we have $[\hat{r}, \hat{P}^r] = \mathrm{i}\hbar\hat{r}$, the Sack dilation algebra [10]. Using the CBH expansion we can easily show

$$\exp(\mathrm{i}\varsigma\hat{P}_r/\hbar)\hat{r}\exp(-\mathrm{i}\varsigma\hat{P}_r/\hbar) = \hat{r}\mathrm{e}^{\varsigma}. \tag{8}$$

To explicitly construct a $\hat{P}^r$ we form

$$\hat{P}^r \equiv \hat{r}\hat{P}^D - \frac{\mathrm{i}\hbar}{2} \tag{9}$$

where the additional $\hbar/2$ is needed for the Hermitian properties of the operator $\hat{P}^r$, as we will see below.

To check that this operator is self-adjoint we first obtain the representation of $\hat{P}^r$ in the $|r\rangle$ basis to find

$$\langle r|\hat{P}^r|\psi\rangle = -\mathrm{i}\hbar(r\partial_r + 1)\langle r|\psi\rangle. \tag{10}$$

As above, we must now find the subclass of all normalizable wavefunctions for which $\hat{P}^r$ is an Hermitian operator. Examining the condition $(\hat{P}^r\psi, \phi) = (\psi, \hat{P}^r\phi)$, in the re-scaled wavefunctions, $U \equiv \phi/r$, and $V \equiv \psi/r$, we find that the wavefunctions $U(r)$, and $V(R)$, must satisfy, $V^*U|_0^\infty = 0$. From $U, V \in \mathcal{L}^2(\mathbb{R}^+, \mathrm{d}r/r)$, we see that $U, V = 0$ at $r = \infty$. Thus the operator $\hat{P}^r$, when restricted to

$$\{\psi(r) \in \mathcal{L}^2(\mathbb{R}^+, r\,\mathrm{d}r), \psi \equiv U/r, U(r = 0) = 0\}$$

is Hermitian. To see whether this operator is also self-adjoint on this class of wavefunctions we must obtain its deficiency indices, i.e. determine the dimensions of the normalizable solution spaces of the equation, $\hat{P}^r\psi = \pm\mathrm{i}\psi$. This again, is most readily achieved by going to the rescaled wavefunction, $V = \psi/r$, where we now have

$$-\mathrm{i}\hbar r\frac{\mathrm{d}}{\mathrm{d}r}V(r) = \pm\mathrm{i}V(r). \tag{11}$$

This equation has the two solutions $V(r) = V_{0\mp}r^{\mp 1/\hbar}$, neither of which are in $\mathcal{L}^2(\mathbb{R}^+, \mathrm{d}r/r)$. Thus for this operator, the deficiency indices are, $(n_+, n_-) = (0, 0)$, and the operator $\hat{P}^r$, acting on this subclass of wavefunctions is *essentially self-adjoint*. The situation here is almost identical to the usual momentum operator ,$-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$, acting on wavefunctions in $\mathcal{L}^2(R, \mathrm{d}x)$ [11]. Thus we have shown that Dirac's Radial momentum, $\hat{P}^D$, is not self-adjoint and does not possess a self-adjoint extension while the new radial momentum, $\hat{P}^r$, is an essentially self-adjoint Hermitian operator on $\mathcal{L}^2(\mathbb{R}^+, r\,\mathrm{d}r)$.

We now continue to explore the properties of this new momentum operator.

From (10) we can calculate the transition function $\langle r|P^r\rangle$ where $|P^r\rangle$ is the eigen-ket of the operator $\hat{P}^r$ to be

$$\langle r|P^r\rangle = \frac{1}{\sqrt{2\pi\hbar}}\frac{r^{\mathrm{i}P^r/\hbar}}{r}. \tag{12}$$

This is normalized with the measure $r\,\mathrm{d}r$ and gives

$$\int_0^\infty r\,\mathrm{d}r\langle r|P^r\rangle\langle P'^r|r\rangle = \delta(P^r - P'^r).$$

The action of $\hat{r}$ on $\hat{P}^r$ is not a simple scaling, instead

$$\mathrm{e}^{-\mathrm{i}\varsigma\hat{r}/\hbar}\hat{P}^r\mathrm{e}^{\mathrm{i}\varsigma\hat{r}/\hbar} = \hat{P}^r + \varsigma\hat{r}.$$

To form a displacement, or point, operator we might be tempted to exponentiate a linear combination of the position and momentum operators in analogy with the harmonic oscillator. However, this construction does not result in a unique adjoint action of the displacement operator on $\hat{r}$ and $\hat{P}^r$. From [13] we see

$$\begin{aligned}\mathcal{D}(a, m) &\equiv \exp\left(\frac{\mathrm{i}a}{\hbar}[\hat{P}^r + m\hat{r}]\right)\\ &= \exp\left(\frac{\mathrm{i}a}{\hbar}\hat{P}^r\right)\exp\left[\frac{\mathrm{i}m\hat{r}}{\hbar}(1 - \mathrm{e}^{-a})\right]\\ &= \exp\left[\frac{\mathrm{i}m\hat{r}}{\hbar}(\mathrm{e}^{a} - 1)\right]\exp\left(\frac{\mathrm{i}a}{\hbar}\hat{P}^r\right).\end{aligned} \tag{13}$$

Thus the choice of ordering alters the action of $\mathcal{D}(a, m)$ on the operators $\hat{r}$ and $\hat{P}^r$. This is to be expected since the algebra is not symmetric in $\hat{r}$ and $\hat{P}^r$. One must explicitly specify the ordering in this displacement operator as is done in most treatments of displacement operators for spin systems.

An essential property for a displacement operator is [6, 14],

$$\mathrm{Tr}[D(\lambda, \mu)D^\dagger(\lambda', \mu')] \approx \delta(\lambda - \lambda')\delta(\mu - \mu'). \tag{14}$$

No operator in the form of products of exponentials of $\hat{r}$ and $\hat{P}^r$ will satisfy (14). We instead consider the operator [16],

$$D(\lambda, \mu) \equiv \exp(\mathrm{i}\mu \hat{P}^r/2\hbar) r^{\mathrm{i}\lambda} \exp(\mathrm{i}\mu \hat{P}^r/2\hbar) \tag{15}$$

where $\lambda \in [0, +\infty)$. Taking the trace over $|r\rangle$ with measure $\frac{r\mathrm{d}r}{2\pi}$, from $r = 0$ to $r = +\infty$ gives

$$\mathrm{Tr}[D(\lambda, \mu)D^\dagger(\lambda', \mu')] = \frac{1}{\mathrm{e}^{2\mu}}\delta(\lambda - \lambda')\delta(\mu - \mu'). \tag{16}$$

One can show $D^\dagger(\lambda, \mu)\hat{r}D(\lambda, \mu) = \hat{r}\mathrm{e}^{-\mu}$. However, to evaluate $D^\dagger(\lambda, \mu)\hat{P}^r D(\lambda, \mu)$ one must calculate $\hat{r}^\alpha \hat{P}^r \hat{r}^{-\alpha}$. This is done by expressing $\hat{P}^r$ in terms of $\hat{P}^D$ through (9), using the simpler commutation relations for $\hat{P}^D$, the rule $[f(\hat{r}), \hat{P}^D] = \mathrm{i}\hbar\partial f(\hat{r})/\partial\hat{r}$, and finally transforming back to $\hat{P}^r$. This gives $D^\dagger(\lambda, \mu)\hat{P}^r D(\lambda, \mu) = \hat{P}^r + \lambda$.

Thus, by using the new 'displacement' operator (15), one can (except for the exponential prefactor in (16)), almost completely regain the properties of the harmonic oscillator displacement operator. Although one could now proceed to construct a Wigner function with this displacement operator, the replacement of $\exp(\mathrm{i}\lambda\hat{r})$ with $\hat{r}^{\mathrm{i}\lambda}$ in (15), and the exponential prefactor in (16) point towards a clearer understanding of the situation.

Guided by these signals we now make the transformation to a new coordinate operator,

$$\hat{v} \equiv \ln \hat{r}. \tag{17}$$

This is a well-defined operation as the unbounded self-adjoint operator $\hat{r}$ possesses a real non-negative spectrum. The transformation (17) has previously been used in WKB studies of radial dynamics [17, 18], and is known there as the Langer transformation. Transformation (17) has also appeared in the geometric quantization of motion on the half-line†.

Using the definition of $\hat{P}^r$ in terms of $\hat{P}^D$ (9), we can show

$$[\hat{v}, \hat{P}^r] = [\ln \hat{r}, \hat{P}^r] = [\ln \hat{r}, \hat{r}\hat{P}^D] = \hat{r}[\ln \hat{r}, \hat{P}^D] = \hat{r}\frac{\mathrm{i}\hbar}{\hat{r}} = \mathrm{i}\hbar \tag{18}$$

the standard Heisenberg–Weyl algebra. Thus we have regained the algebra of the original Cartesian position and momentum observables. The transformation from $(\hat{r}, \hat{P}^D)$, to the new coordinates $(\hat{v}, \hat{P}^r)$, is unitary while the corresponding classical transformation is canonical. Since the spectrum for both $\hat{v}$ and $\hat{P}^r$ ranges from $(-\infty, +\infty)$ the translation generating Weyl algebra in (18) properly respects the spectrum of both operators with the result that both $\hat{v}$ and $\hat{P}^r$ are self-adjoint on the domain of normalizable wavefunctions and thus represent physical observables.

Since we now have a Heisenberg–Weyl algebra, we can construct creation and annihilation operators in the same manner as the harmonic oscillator,

$$\begin{aligned} \sqrt{2}\hat{a} &\equiv \hat{v} + \mathrm{i}\hat{P}^r \\ \sqrt{2}\hat{a}^\dagger &\equiv \hat{v} - \mathrm{i}\hat{P}^r. \end{aligned} \tag{19}$$

† See [5], equation 4.5.56. Note that the dimensionful quantities in $\hat{r}$ drop out when we take the logarithm.

In particular, there exists a proper vacuum state $|0\rangle_{\rm dilations}$ which is annihilated by $\hat{a}$.

Before we can construct the point operators and the Wigner function we must first address the non-trivial measure $r\mathrm{d}r$, and how this appears in the new basis $(\hat{v}, \hat{P}^r)$. The resolution of unity in the $|r\rangle$ basis becomes, for cylindrical symmetry, in the eigenbasis of $\hat{v}$,

$$\mathbb{1} = \int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{d}y|x, y\rangle\langle x, y| = 2\pi \int_0^{\infty} r\,\mathrm{d}r|r\rangle\langle r| \tag{20}$$

$$= 2\pi \int_{-\infty}^{+\infty} \mathrm{e}^{2v}\,\mathrm{d}v|v\rangle\langle v|. \tag{21}$$

With this resolution of unity we must have

$$\mathbb{1}|v'\rangle = |v'\rangle = 2\pi \int_{-\infty}^{+\infty} \mathrm{e}^{2v}\,\mathrm{d}v|v\rangle\langle v|v'\rangle \tag{22}$$

and thus $\langle v|v'\rangle \equiv \frac{1}{\mathrm{e}^{2v}}\delta(v - v') = 2\delta(\mathrm{e}^{2v} - \mathrm{e}^{2v'})$. As is usually done in the treatment of radial quantum systems [17], one can shift the influence of the non-trivial measure into the wavefunction through a redefinition of the states. This gives a simpler measure at the expense of altering the form of any dynamical equations which these states might obey [17]. We can thus rescale the basis kets to be

$$|\bar{v}\rangle^{\bullet} \equiv \mathrm{e}^{\bar{v}}|\mathrm{e}^{\bar{v}}\rangle_r \tag{23}$$

where $|\mathrm{e}^{\bar{v}}\rangle_r$ is an eigenket of $\hat{r}$ with eigenvalue $\mathrm{e}^{\bar{v}}$. The resolution of the identity in the new $|\bar{v}\rangle^{\bullet}$ basis is,

$$\mathbb{1} = 4\pi \int_{-\infty}^{+\infty} \mathrm{d}\bar{v}|\bar{v}\rangle^{\bullet\bullet}\langle \bar{v}| \tag{24}$$

where the inner product between $|\bar{v}\rangle^{\bullet}$ basis kets is now simply ${}^{\bullet}\langle\bar{v}|\bar{v}'\rangle^{\bullet} \equiv \delta(\bar{v} - \bar{v}')$. This rescaling will primarily appear in the calculation of ${}^{\bullet}\langle\bar{v}|\psi\rangle$ when we have $\langle r|\psi\rangle$. To be certain that everything is identical to the familiar Heisenberg–Weyl algebra associated with quantum mechanics on the fully infinite line (i.e. $\psi \in L^2(\mathbb{R}, \mathrm{d}x)$), we finally evaluate ${}^{\bullet}\langle\bar{v}|\hat{P}^r|\psi\rangle$. Since $|\bar{v}\rangle^{\bullet} = r|r\rangle$ where $r = \mathrm{e}^{\bar{v}}$, we have

$$\begin{aligned}
{}^{\bullet}\langle\bar{v}|\hat{P}^r|\psi\rangle &= r\langle r|\hat{P}^r|\psi\rangle|_{r=\mathrm{e}^{\bar{v}}} = r[-\mathrm{i}\hbar(r\partial_r + 1)]\langle r|\psi\rangle|_{r=\mathrm{e}^{\bar{v}}} \\
&= r\left[-\mathrm{i}\hbar(r\partial_r + 1)\frac{1}{r}\right]{}^{\bullet}\langle\bar{v}|\psi\rangle|_{r=\mathrm{e}^{\bar{v}}} \\
&= -\mathrm{i}\hbar r\partial_r{}^{\bullet}\langle\bar{v}|\psi\rangle|_{r=\mathrm{e}^{\bar{v}}} = -\mathrm{i}\hbar\partial_{\bar{v}}{}^{\bullet}\langle\bar{v}|\psi\rangle.
\end{aligned} \tag{25}$$

We see that the momentum operator $\hat{P}^r$, in the basis $|\bar{v}\rangle^{\bullet}$ is the familiar $-\mathrm{i}\hbar\partial_{\bar{v}}$.

3. Radial dilation Wigner function

As in the standard case of quantum mechanics on the fully infinite line in Cartesian coordinates we define the Wigner and s-ordered quasidistribution functions to be

$$W(\xi, s) \equiv \int \frac{\mathrm{d}^2\alpha}{\pi} \mathrm{e}^{\alpha\xi^* - \alpha^*\xi} C(\alpha, s) \tag{26}$$

where $C(\alpha, s)$ is the s-ordered displacement operator

$$C(\alpha, s) \equiv \mathrm{Tr}[\rho D(\alpha)\mathrm{e}^{s|\alpha|^2/2}] \tag{27}$$

and $D(\alpha) \equiv \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, where $\hat{a}^\dagger$ and $\hat{a}$, the dilation creation and annihilation operators, are defined as above. For $s = 0$, $\xi = \gamma + \mathrm{i}\delta$, the Wigner function becomes

$$W(\gamma,\delta) = \frac{1}{2\pi}\int \mathrm{d}\varepsilon \mathrm{e}^{-\mathrm{i}\varepsilon\delta\bullet}\left\langle\gamma + \frac{\varepsilon}{2}|\rho|\gamma - \frac{\varepsilon}{2}\right\rangle^{\bullet}. \tag{28}$$

We now calculate the Wigner function for the quantum states $\rho_l \equiv |l,0\rangle\langle l,0|$, where $|l,m\rangle$ is the Schwinger angular momentum state given by

$$|l,m\rangle \equiv \frac{1}{\sqrt{(l+m)!(l-m)!}}\hat{A}_+^{\dagger(l+m)}\hat{A}_-^{\dagger(l-m)}|0,0\rangle \tag{29}$$

where $\hat{A}_+ \equiv (\hat{a}_x - \mathrm{i}\hat{a}_y)/\sqrt{2}$, $\hat{A}_- \equiv (\hat{a}_x + \mathrm{i}\hat{a}_y)/\sqrt{2}$ [19]. Either by solving the generating differential equation (as is done in [19]) or by solving the harmonic oscillator radial equation one finds $\langle r,\phi|l,m\rangle = R_{l,m}(r)\Theta_m(\phi)$, where

$$R_{l,m}(r) = \beta\sqrt{\frac{2(l-|m|)!}{(l+|m|)!}}(\beta r)^{2|m|}\mathrm{e}^{-\beta^2r^2/2}\mathrm{L}_{l-|m|}^{2|m|}(\beta^2r^2)(-1)^{l-|m|} \tag{30}$$

$$\Theta_m(\phi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{2\mathrm{i}m\phi} \tag{31}$$

L_n^α is the associated Laguerre polynomial and $\beta = 1/\sqrt{\hbar}$. Integrating out the ϕ dependence, setting $\hbar = 1$ and using (23) we obtain

$$^\bullet\langle\bar{v}|l,0\rangle = \sqrt{2}\mathrm{e}^{\bar{v}}\mathrm{e}^{-\mathrm{e}^{2\bar{v}}/2}\mathrm{L}_l^0(\mathrm{e}^{2\bar{v}})(-1)^l. \tag{32}$$

Inserting (32) into (28) we finally find

$$W_l(\gamma,\delta) = \frac{2\mathrm{e}^{2\gamma}}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}\varepsilon\mathrm{e}^{-2\mathrm{i}\varepsilon\delta}\exp[-\mathrm{e}^{2\gamma}\cosh 2\varepsilon]\mathrm{L}_l(\mathrm{e}^{2(\gamma+\varepsilon)})\mathrm{L}_l(\mathrm{e}^{2(\gamma-\varepsilon)}). \tag{33}$$

The radial Wigner function, (33), is numerically integrated for the first four $|l,0\rangle$ states and plotted in figures 1(*a*)–(*d*). The presence of negative regions in these Wigner functions is not surprising as the pure states $|l,0\rangle$ are reminiscent of Fock states. Further, one can construct the dilation coherent states $|\alpha\rangle \equiv D(\alpha)|0\rangle_{\mathrm{dilation}}$. More interesting, however, is the dilation vacuum state, $\langle r|0\rangle_{\mathrm{dilation}}$. From the definition $\hat{a}|0\rangle_{\mathrm{dilation}} = 0$, and $\sqrt{2}\hat{a} = \hat{v} + \mathrm{i}\hat{P}^r$, we can evaluate $^\bullet\langle\bar{v}|\hat{a}|0\rangle_{\mathrm{dilation}} = 0$ to get $^\bullet\langle\bar{v}|0\rangle_{\mathrm{dilation}} = N\exp(-\bar{v}^2/2)$, where $N^2 = \sqrt{\pi}$. Again using (23) we obtain

$$\langle r|0\rangle_{\mathrm{dilation}} = \frac{1}{r\sqrt{\pi}}\exp\left(-\frac{1}{2}\ln r\ln r\right) = \frac{r^{-\frac{1}{2}\ln r}}{r\sqrt{\pi}} \tag{34}$$

which is normalized to unity,

$$\int_0^{+\infty} r\,\mathrm{d}r|\langle r|0\rangle_{\mathrm{dilation}}|^2 = 1. \tag{35}$$

4. Conclusion

In this paper we examined the problem of constructing a Wigner function for the two-dimensional radial subspace of a quantum system possessing two continuous degrees of freedom (or a four-dimensional Wigner function). Since the radial subspace is labelled by an operator with an half-infinite spectrum, previous attempts to define a physical conjugate momentum have be suspect. By choosing a momentum operator which respected the spectrum of the radial coordinate we found we could construct a physically meaningful

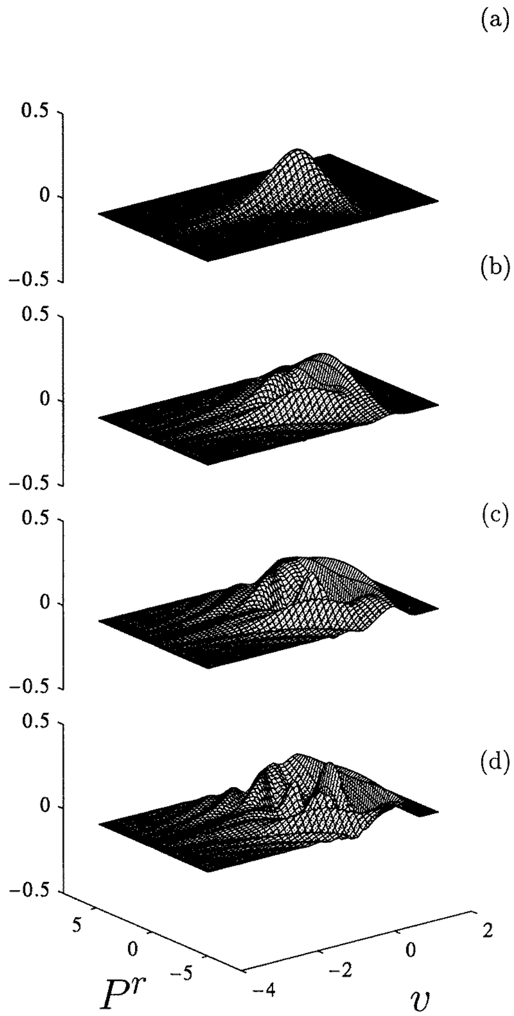

Figure 1. (*a*)–(*d*) Radial Wigner functions for the states $|l, 0\rangle$, $l = 0, \ldots, 3$.

radial conjugate momentum. With a logarithmic relabelling of the radial coordinate the new coordinate and momentum satisfied the Heisenberg–Weyl algebra and we were thus able to carry over all the techniques associated with this algebra to construct a Wigner function. We identified a ground state for these observables which was destroyed by a dilation annihilation operator. We finally examined the radial Wigner function for particular quantum states and found the radial wavefunction for the dilation vacuum state.

To examine the radial Wigner function for more general quantum states is difficult. If one has the density matrix in a $x - y$ Fock state basis, one must first transform ρ into the Schwinger basis of angular momentum kets, $\rho = \sum C_{lm;l'm'}|l, m\rangle\langle l', m'|$. One can then find $\langle r, \phi|\rho|r', \phi'\rangle$. Finally one must trace over ϕ. However, even though we have traced out over ϕ one will still end up with a sum over m. This is because, in contrast with the Cartesian Fock state decomposition, the radial wavefunction is labelled by *both* quantum numbers l and m while the angular part is only labelled by m. We also note that the Wigner function here gives the correct marginals for the operators $\hat{P}^r$ and $\hat{v}$. Once these marginals and their corresponding integration measures are obtained one can rescale v to r. One cannot rescale the axis in figure 1 to find the Wigner function for $\hat{P}^r$ and $\hat{r}$ without altering the integration measure and thus, the functional form of the Wigner function. The reasoning used to arrive at the self-adjoint conjugate momentum $\hat{P}^r$ may be applicable to systems with more complicated boundary conditions, i.e. $\psi \in L^2([0, L_b(t)], \mathrm{d}x)$. This will be reported in a later work. Finally, although an observable, the radial operator $\hat{v} = \ln \hat{r}$, may not be easy to measure directly. However, it should be possible, through standard reconstruction techniques, to numerically approximate the radial Wigner function from experimental data.

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